Binomial Theorem

1. $(10.1)^5 = (10 + 0.1)^5 = 10^5 + 5(10)^4(0.1) + 10(10)^3(0.1)^2 + 10(10)^2(0.1)^3 + 5(10)(0.01)^4 + (0.1)^5$

$= 100000 + 5000 + 100 + 1 + 0.005 + 0.00001 = 105101.00501$

2. $\left(2+\sqrt{3}\right)^4 + \left(2-\sqrt{3}\right)^4 = 2\left[2^4 + 6(2)^2\left(\sqrt{3}\right)^2 + \left(\sqrt{3}\right)^4\right] = 194$

$0 < 2-\sqrt{3} < 1 \Rightarrow 0 < \left(2-\sqrt{3}\right)^4 < 1 \Rightarrow 193 < \left(2+\sqrt{3}\right)^4 = 194 - \left(2-\sqrt{3}\right)^4 < 194.$

3. **(a)** $(1 + ax + bx^2)^6 = [1 + x(a + bx)]^6 = 1 + 6x(a+bx) + 15x^2(a+bx)^2 + \ldots$

$= 1 + 6ax + 6bx^2 + 15x^2 (a^2 + 2bx + b^2x^2) + \ldots$

$= 1 - 12x + 78x^2 + \ldots \qquad \therefore\ a = -2, \quad b = 3.$

(b) $T_{r+1} = C_r^9\left(3x^2\right)^{9-r}\left(-\frac{1}{2x}\right)^r = C_r^9(3)^{9-r}\left(-\frac{1}{2}\right)^r x^{18-3r}$

$T_{r+1} =$ is independent of $x \Rightarrow 18 - 3r = 0 \Rightarrow r = 6 \qquad \therefore\ T_7 = C_6^9 3^{9-6}\left(-\frac{1}{2}\right)^6 = \frac{567}{16}$

4. **(a)** $\frac{T_{r+1}}{T_r} = \frac{C_r^8(4x)^r}{C_{r-1}^8(4x)^{r-1}} = \frac{9-r}{r}(4x) = \frac{9-r}{r}\left(\frac{4}{3}\right) \ge 1 \Rightarrow \quad r \le 5\frac{1}{7}$

$\therefore$ Greatest term $= T_6 = C_5^8\left(\frac{4}{3}\right)^5 = \frac{57344}{243} = 235\frac{239}{243}$

(b) $\frac{T_{r+1}}{T_r} = \frac{C_r^9(2x)^r 3^{9-r}}{C_{r-1}^9(2x)^{r-1}3^{9-r+1}} = \frac{10-r}{r}\left(\frac{2x}{3}\right) = \frac{10-r}{r}\left(\frac{2}{3}\right) \ge 1 \Rightarrow \quad r \le 4$

$\therefore$ Greatest term $= T_4$ or $T_5 = C_4^9 2^4 3^5 = 489888$

5. Coefficient of x^n in $(1 + x)^{2n} = C_n^{2n} = \frac{(2n)!}{n!n!}$

Coefficient of x^n in $(1 + x)^{2n-1} = C_n^{2n-1} = \frac{(2n-1)!}{n!(2n-1-n)!} = \frac{(2n-1)!}{n!(n-1)!} = \frac{1}{2}\frac{(2n)\times(2n-1)!}{n![n\times(n-1)!]} = \frac{1}{2}\frac{(2n)!}{n!n!}$

Result follows.

6. Coeff. of x^m in $(1 + x)^{m+n} = C_m^{m+n} = C_{(m+n)-m}^{m+n} = C_n^{m+n}$ = Coeff. of x^n in $(1 + x)^{m+n}$

7. $(3 + 2x - x^2)(1 + x)^{34} = (3 - x)(1 + x)(1 + x)^{34} = (3 - x)(1 + x)^{35} = (3-x)\left(\sum_{r=0}^{35} C_r^{35}x^r\right)$

Coeff. of $x^r = 3C_r^{35} - C_{r-1}^{35} = 0 \Rightarrow \frac{C_{r-1}^{35}}{C_r^{35}} = 3 \Rightarrow \frac{r}{36-r} = 3 \Rightarrow r = 27$

8. $\frac{C_r^n}{C_{r-1}^n} = \frac{n-r+1}{r} \ge 1 \Leftrightarrow r \le \frac{n+1}{2}$

(i) If $n = 2p$ is even, the greatest coeff. is $C^n_{n/2} = C^{2p}_p = \frac{(2p)!}{p!p!} = \frac{(2p)(2p-1)...3.2.1}{p!p!}$

$$= \frac{(2p-1)(2p-3)...5.3.1}{p!p!}[p(p-1)...2.1]2^p = \frac{(2p-1)(2p-3)...5.3.1}{p!}2^p = \frac{1\cdot 3\cdot 5\cdots(n-1)}{1\cdot 2\cdot 3\cdots\left(\frac{n}{2}\right)}2^{\frac{n}{2}}$$

(ii) If $n = 2p+1$ is odd, the greatest coefficients are $C^n_{(n-1)/2}$ or $C^n_{(n+1)/2} = C^{2p+1}_{p-1}$ or C^{2p+1}_{p+1}

$$= \frac{(2p+1)!}{p!(p+1)!} = \frac{(2p+1)(2p-1)...5.3.1[p(p-1)...2.1.2^p]}{p!(p+1)!} = \frac{1.3.5...(2p+1)}{(p+1)!}2^p = \frac{1\cdot 3\cdot 5\cdots n}{1\cdot 2\cdot 3\cdots\left(\frac{n+1}{2}\right)}2^{\frac{n-1}{2}}$$

9. $f(x) = (1 + x)^n = c_0 + c_1x + c_2x^2 + \dots + c_nx^n$

$f'(x) = n(1 + x)^{n-1} = c_1 + 2c_2x + 3c_3x^2 + \dots + nc_nx^{n-1}$

$f'(-1) = 0 = c_1 - 2c_2 + 3c_3 - \dots + n(-1)^{n-1}c_n$.

10. $(1 + x)^n = c_0 + c_1x + c_2x^2 + \dots + c_nx^n$. and $(x + 1)^n = c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \dots + c_n$.

Since $(1 + x)^{2n} = (1 + x)^n (x + 1)^n$

Compare coeff. of x^{n+r} in the expansion of $(1 + x)^{2n}$,

$$c_0c_r + c_1c_{r+1} + c_2c_{r+2} + \dots + c_{n-r}c_n = C^{2n}_{n+r} = \frac{(2n)!}{(n+r)!(n-r)!} = \frac{(2n)(2n-1)....(n-r+1)}{(n+r)!}$$

11. Compare coeff. of x^{n+1} in the expansion of No. 10,

$$c_0c_1 + c_1c_2 + \dots + c_{n-1}c_n = \sum_{r=0}^{n-1} c_rc_{r+1} = C^{2n}_{n+1} = \frac{(2n)!}{(n+1)!(n-1)!} = \frac{(2n)\times(2n-1)!}{(n+1)n\times(n-1)!(n-1)!} = \frac{2(2n-1)!}{(n+1)[(n-1)!]^2}$$

12. $(1 + x)^n = c_0 + c_1x + c_2x^2 + \dots + c_nx^n$. $\therefore$ $\frac{(1+x)^n - 1}{x} = c_1 + c_2x + \dots + c_nx^{n-1}$

Differentiate, we get $\frac{nx(1+x)^{n-1} - [(1+x)^n - 1]}{x^2} = c_2 + 2c_3x + \dots + (n-1)c_nx^{n-2}$.

Put $x = 1$, $n2^{n-1} - 2^n + 1 = c_2 + 2c_3x + \dots + (n-1)c_nx^{n-2}$, $\therefore \sum_{r=0}^{n-1} rc_{r+1} = 1 + (n-2)2^{n-1}$

13. **(a)** ${}_{n+1}C_r - {}_nC_{r-1} = \frac{(n+1)!}{r!(n+1-r)!} - \frac{n!}{(r-1)!(n-r+1)!} = \frac{n!}{r!(n-r+1)!}[(n+1)-r] = \frac{n!}{r!(n-r)!} = {}_nC_r$.

(b) Replace n by 2n and r by n in (a), ${}_{2n+1}C_n - {}_{2n}C_{n-1} = {}_{2n}C_n$ ….(1)

$$(n+1)[{}_{2n+1}C_n + {}_{2n}C_{n-1}] = (n+1)\left[\frac{(2n+1)!}{n!(n+1)!} + \frac{(2n)!}{(n-1)!(n+1)!}\right] = (n+1)\frac{(2n)!}{n!(n+1)!}[(2n+1)+n] = (3n+1)\frac{(2n)!}{n!n!}$$

$= (3n+1)\,{}_{2n}C_n$ ….(2)

(1) × (2), $(3n + 1)\,[{}_{2n}C_n]^2 = (n + 1)\,\{[{}_{2n+1}C_n]^2 - [{}_{2n}C_{n-1}]^2\}$.

14. $(1 + x)^{2n} = c_0 + c_1x + c_2x^2 + \dots + c_{2n}x^{2n}$,

Put $x = 1$, $2^{2n} = c_0 + c_1 + c_2 + \dots + c_{2n}$ ….(1)

Put $x = -1$, $0 = c_0 - c_1 + c_2 - \dots + c_{2n}$ ….(2)

[(1) + (2)]/2, $2^{2n-1} = c_0 + c_2 + c_4 + \dots + c_{2n}$.

15. $(1 + x + x^2)^{10} = c_0 + c_1x + c_2x^2 + \ldots + c_{20}x^{20}$,

Put $x = 1$, $c_0 + c_1 + c_2 + \ldots + c_{20} = 3^{10}$ ….(1)

Put $x = -1$, $c_0 - c_1 + c_2 - \ldots + c_{20} = 1$ ….(2)

$\frac{(1)+(2)}{2}$, $c_0 + c_2 + c_4 + \ldots + c_{20} = \frac{3^{10}+1}{2} \Rightarrow c_2 + c_4 + \ldots + c_{20} = \frac{3^{10}-1}{2}$, $\because c_0 = 1$

$\frac{(1)-(2)}{2}$, $c_1 + c_3 + c_5 + \ldots + c_{19} = \frac{3^{10}-1}{2}$. Result follows.

16. $(1 + x)^{2m+1} = c_0 + c_1x + c_2x^2 + \ldots + c_{2m+1}x^{2m+1}$. Put $x = 1$, $c_0 + c_1 + c_2 + \ldots + c_{2m+1} = 2^{2m+1}$.

But $c_0 = c_{2m+1}$, $c_1 = c_{2m}$, …., $c_m = c_{m+1}$, $2(c_0 + c_1 + c_2 + \ldots + c_m) = 2^{2m+1}$. Result follows.

17. $(y+a)^n = \sum_{r=0}^{n} C_r^n a^r y^{n-r}$. Put $y = x - a$, $x^n = \sum_{r=0}^{n} C_r^n a^r (x-a)^{n-r} \Rightarrow p_r = ({}_nC_r)\, a^r$.

18. $(1 + x)^n + (1 + x)^{n+1} + \ldots + (1 + x)^{n+k} = (1+x)^n \frac{(1+x)^{k+1}-1}{(1+x)-1} = \frac{(1+x)^{n+k+1}-(1+x)^n}{x}$, Geometric Progression

Consider the x^n – term on both sides, ${}_nC_n + {}_{n+1}C_n + {}_{n+2}C_n + \ldots + {}_{n+k}C_n = {}_{n+k+1}C_{n+1}$.

19. $(1+x)^n = \sum_{r=0}^{n} C_r^n x^r$. Put $x = 1$, $2^n = \sum_{r=0}^{n} C_r^n \Rightarrow 2^n - 2 = \sum_{r=1}^{n-1} C_r^n = \sum_{r=1}^{n-1} \frac{n(n-1)\ldots(n-r+1)}{r!}, C_0^n = C_n^n = 1$

$\therefore$ $(n-1)!(2^n - 2) = \sum_{r=1}^{n-1} n(n-1)\ldots(n-r+1)\left[\frac{(n-1)!}{r!}\right]$

In the summation $\left[\frac{(n-1)!}{r!}\right]$ is an integer since r ranges from 1 to $(n - 1)$ and also each term has a factor n.

$\therefore$ $(n - 1)!\,(2^n - 2) = [2(n - 1)!]\,(2^{n-1} - 1)$ is divisible by n.

If n is a prime number, $[2(n - 1)!]$ is not divisble by n. $\therefore$ $(2^{n-1} - 1)$ is divisible by n.

20. **(a)** $(1 + x)^n (1 + x)^n = (1 + x)^{2n}$

$(c_0 + c_1x + c_2x^2 + \ldots + c_nx^n)(c_0 + c_1x + c_2x^2 + \ldots + c_nx^n) = \sum_{r=0}^{2n} C_r^{2n} x^r$

Compare coeff. of x^r – term, $f(r) = c_0c_r + c_1c_{r-1} + c_2c_{r-2} + \ldots + c_rc_0 =$.

(b) From (a), $(1 + x)^{2n} = f(0) + f(1)\, x + f(2)\, x^2 + \ldots + f(n + 1)\, x^{n+1} + \ldots + f(2n)x^{2n}$ ….(1)

Differentiate (1), $(2n)(1 + x)^{2n-1} = f(1) + 2f(2)\, x + 3f(3)\, x^2 + \ldots + (2n)f(2n)x^{2n-1}$ ….(2)

$(1 + x)^n = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$ ….(3)

$(2) \times (3)$, Coeff. of x^n term on R.H.S. $= c_0\, f(1) + 2c_1\, f(2) + 3c_2\, f(3) + \ldots + (n+1)c_n\, f(n+1)$

Coeff. of x^n term on L.H.S. = .

21. **(a)** $(1 + x)^n (1 + x)^n = (1 + x)^{2n}$

$({}_nC_0 + {}_nC_1\, x + {}_nC_2\, x^2 + \ldots + {}_nC_n\, x^n)({}_nC_0\, x^n + {}_nC_1\, x^{n-1} + \ldots + {}_nC_n) = {}_{2n}C_0 + {}_{2n}C_1x + \ldots + {}_{2n}C_nx^{2n}$

Compare coefficients of x^{2n} -term, we get $({}_nC_0)^2 + ({}_nC_1)^2 + ({}_nC_2)^2 + \ldots + ({}_nC_n)^2 = {}_{2n}C_n$.

(b) $(1-x)^{2n}(1+x)^{2n} = (1-x^2)^{2n}$

$({}_{2n}C_0 - {}_{2n}C_1x + {}_{2n}C_2x^2 - \dots + {}_{2n}C_{2n}\,x^{2n})\,({}_{2n}C_0\,x^{2n} + {}_{2n}C_1x^{2n-1} + \dots + {}_{2n}C_{2n})$

$= {}_{2n}C_0 - {}_{2n}C_1x^2 + {}_{2n}C_2x^4 - \dots + (-1)^n\,{}_{2n}C_n\,x^{2n} + \dots + {}_{2n}C_{2n}\,x^{4n}$.

Compare coefficients of x^{2n} -term, we get $({}_{2n}C_0)^2 - ({}_{2n}C_1)^2 + ({}_{2n}C_2)^2 + \dots - ({}_{2n}C_{2n})^2 = (-1)^n\,{}_{2n}C_n$.

(c) $(1-x)^{2n+1}(1+x)^{2n+1} = (1-x^2)^{2n+1}$

$({}_{2n+1}C_0 - {}_{2n+1}C_1x + {}_{2n+1}C_2x^2 - \dots + {}_{2n+1}C_{2n+1}\,x^{2n+1})\,({}_{2n+1}C_0\,x^{2n+1} + {}_{2n+1}C_1x^{2n} + \dots + {}_{2n+1}C_{2n+1})$

$= {}_{2n+1}C_0 - {}_{2n+1}C_1x^2 + {}_{2n+1}C_2x^4 - \dots + 0 \times x^{2n+1} + \dots + {}_{2n+1}C_{2n+1}\,x^{4n+2}$.

Compare coefficients of x^{2n+1} -term, we get $({}_{2n+1}C_0)^2 - ({}_{2n+1}C_1)^2 + ({}_{2n+1}C_2)^2 + \dots - ({}_{2n+1}C_{2n+1})^2 = 0$.

(d) $(1+x)^n = {}_nC_0 + {}_nC_1\,x + {}_nC_2\,x^2 + \dots + {}_nC_n\,x^n$ …. (1)

Differentiate (1), $n(1+x)^{n-1} = {}_nC_1 + 2\,{}_nC_2\,x + 3\,{}_nC_3\,x^2 + \dots + n\,{}_nC_n\,x^{n-1}$ …. (2)

Since $n(1+x)^{n-1}(1+x)^n = n(1+x)^{2n-1}$, we have :

$({}_nC_1 + 2\,{}_nC_2\,x + 3\,{}_nC_3\,x^2 + \dots + n\,{}_nC_n\,x^{n-1})({}_nC_0\,x^n + {}_nC_1\,x^{n-1} + \dots + {}_nC_n)$

$= n\,({}_{2n-1}C_0 + {}_{2n-1}C_1x + \dots + {}_{2n-1}C_{n-1}x^{n-1} + \dots + {}_{2n-1}C_{2n-1}\,x^{2n-1})$

Compare coefficients of x^{n-1} -term, we get $({}_nC_1)^2 + 2({}_nC_2)^2 + \dots + n({}_nC_n)^2 = n\,{}_{2n-1}C_{n-1} =$.

22. From 21. (d) eqs. (1) and (2), put $x = 1$, we get :

${}_nC_0 + {}_nC_1 + {}_nC_2 + \dots + {}_nC_n = 2^n$ …. (1)

${}_nC_1 + 2\,{}_nC_2 + 3\,{}_nC_3 + \dots + n\,{}_nC_n = n\,2^{n-1}$ …. (2)

$a \times (1) + b \times (2)$, $a + n(a+b) + {}_nC_2(a+2b) + {}_nC_3(a+3b) + \dots + {}_nC_n\,[a+nb] = 2^{n-1}(2a+nb)$.

23. **(a)** $(1+x)^{2n} + x(1+x)^{2n-1} + \dots + x^n(1+x)^n + x^{n+1}(1+x)^{n-1} + \dots + x^{2n} = (1+x)^{2n}\dfrac{1-\left(\dfrac{x}{1+x}\right)^{2n+1}}{1-\dfrac{x}{1+x}}$

$= (1+x)^{2n+1} - x^{2n+1}$.

(b) Compare coefficients of x^n –term , we get ${}_{2n}C_n + {}_{2n-1}C_{n-1} + \dots + {}_nC_0 = {}_{2n+1}C_n$.

24. $(1+x)^n = {}_nC_0 + {}_nC_1\,x + {}_nC_2\,x^2 + \dots + {}_nC_n\,x^n$ …. (1)

Differentiate (1), $n(1+x)^{n-1} = {}_nC_1 + 2\,{}_nC_2\,x + 3\,{}_nC_3\,x^2 + \dots + n\,{}_nC_n\,x^{n-1}$ …. (2)

Method 1

$(2) \times x$, $nx(1+x)^{n-1} = {}_nC_1\,x + 2\,{}_nC_2\,x^2 + 3\,{}_nC_3\,x^3 + \dots + n\,{}_nC_n\,x^n$ …. (3)

Differentiate (3), $n(n-1)x(1+x)^{n-2} + n(1+x)^{n-1} = {}_nC_1 + 2^2\,{}_nC_2\,x + 3^2\,{}_nC_3\,x^2 + \dots + n^2\,{}_nC_n\,x^{n-1}$ …. (4)

Put $x = 1$ in (4), ${}_nC_1\,1^2 + {}_nC_2\,2^2 + \dots + {}_nC_n\,n^2 = n(n-1)\,2^{n-2} + n2^{n-1} = 2^{n-2}(n^2 - n + 2n) = n(n+1)2^{n-2}$.

Method 2

Differentiate (2), $n(n-1)(1+x)^{n-2} + n(1+x)^{n-1} = (2)(1)\,{}_nC_2\,x + (3)(2)\,{}_nC_3\,x^2 + \dots + n(n-1)\,{}_nC_n\,x^{n-1}$ …. (5)

Put $x = 1$ in (2), $n\,2^{n-1} = {}_nC_1 + 2\,{}_nC_2 + 3\,{}_nC_3 + \dots + n\,{}_nC_n$ …. (6)

Put $x = 1$ in (5), $n(n-1)2^{n-2} + n\,2^{n-1} = (2)(1)\,{}_nC_2 + (3)(2)\,{}_nC_3 + \dots + n(n-1)\,{}_nC_n$ …. (7)

(6) + (7), using ${}_nC_k\,k^2 = k(k-1)\,{}_nC_k + k\,{}_nC_k$ for RHS, result follows.

25. $(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \dots + a_{2n}x^{2n}$ …. (1)

(a) Put $x = 1$ in (1), $a_0 + a_1 + \dots + a_{2n} = (1 + 1 + 1^2)^n = 3^n$.

(b) Put $x = -1$ in (1), $a_0 - a_1 + a_2 - a_3 + \dots + a_{2n} = (1 - 1 + 1^2)^n = 1$.

(c) Replace x by 1/x in (1),

$[1 + (1/x) + (1/x^2)]^n = a_0 + a_1 (1/x) + a_2 (1/x^2) + \dots + a_{2n} (1/x^{2n})$ …. (2)

$(2) \times x^{2n}$, $(1 + x + x^2)^n = a_{2n} + a_{2n-1} x + a_{2n-2} x^2 + \dots + a_0 x^{2n}$ …. (3)

Compare coefficients in (1) and (3), $a_k = a_{2n-k}$, $k = 0, 1, \dots, 2n$

or $a_{n-r} = a_{n+r}$, $r = 0, 1, \dots, n$

(d) $(1 + x + x^2)^n (1 - x + x^2)^n = (1 + x^2 + x^4)^n$.

$(a_0 + a_1 x + a_2 x^2 + \dots + a_{2n}x^{2n}) (a_0 - a_1 x + a_2 x^2 - \dots + a_{2n}x^{2n}) = a_0 + a_1 x^2 + a_2 x^4 + \dots + a_{2n}x^{4n}$ …. (4)

Compare coefficients of x^{2n} on both sides, $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \dots + a_{2n}^2 = a_n$.

(e) From (d) and since $a_k = a_{2n-k}$, we have $2[a_0^2 - a_1^2 + a_2^2 - a_3^2 + \dots + (-1)^{n-1} a_{n-1}^2] + (-1)^n a_n^2 = a_n$.

$\therefore$ $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \dots + (-1)^{n-1} a_{n-1}^2 =$

(f) From (4), since $a_k = a_{2n-k}$,

$(a_{2n} + a_{2n-1} x + a_{2n-2} x^2 + \dots + a_0x^{2n}) (a_0 - a_1 x + a_2 x^2 - \dots + a_{2n}x^{2n}) = a_0 + a_1 x^2 + a_2 x^4 + \dots + a_{2n}x^{4n}$ ….(5)

Compare coefficients of $x^{2n+2} = 2^{2(n+1)}$ -term in (5), $a_0a_2 - a_1a_3 + a_2a_4 - a_3a_5 + \dots + a_{2n-2}a_{2n} = a_{n+1}$.

26. **(a)** $(1 + x)^m = {}_mC_0 + {}_nC_1 x + {}_mC_2 x^2 + \dots + {}_mC_m x^m$. Put x = 1, ${}_mC_0 + {}_mC_1 + {}_mC_2 + \dots + {}_mC_m = 2^m$.

(b) In (a), put $m = 2n+1$, ${}_{2n+1}C_1 + {}_{2n+1}C_2 + \dots + {}_{2n+1}C_{2n-1} = 2^m - {}_{2n+1}C_0 - {}_{2n+1}C_{2n+1} = 2^{2n+1} - 2$.

But, ${}_{2n+1}C_r = {}_{2n+1}C_{2n+1-r}$, $\therefore$ $2[{}_{2n+1}C_1 + {}_{2n+1}C_2 + \dots + {}_{2n+1}C_n] = 2^{2n+1} - 2$.

$\therefore$ ${}_{2n+1}C_1 + {}_{2n+1}C_2 + \dots + {}_{2n+1}C_n = 2^{2n} - 1$.

$\therefore$.

27. ${}_nC_{18} = {}_nC_7 \Rightarrow {}_nC_{18} = {}_nC_{n-7} \Rightarrow 18 = n - 7 \Rightarrow n = 25$.

$\therefore$ ${}_nC_{22} = {}_{25}C_{22} = {}_{25}C_3 = \dfrac{25\times 24\times 23}{3\times 2\times 1} = 2300$ and ${}_{27}C_n = {}_{27}C_{25} = {}_{27}C_2 = \dfrac{27\times 26}{2\times 1} = 351$

28. $$\frac{a_1}{a_0}+2\frac{a_2}{a_1}+3\frac{a_3}{a_2}+\dots+n\frac{a_n}{a_{n-1}}=\sum_{r=1}^{n} r\frac{{}_nC_r}{{}_nC_{r-1}}=\sum_{r=1}^{n}(n-r+1)=\sum_{r=1}^{n} r=\frac{1}{2}n(n+1)$$

29. Coeff. of $x^r = {}_nC_r + 2^r\, {}_nC_r + 4^r\, {}_nC_r = (1 + 2^r + 4^r)\, {}_nC_r$.

$$a_3 : a_{n-3} = a_3 : a_6 = \frac{(1+2^3+4^3)\,{}_9C_3}{(1+2^6+4^6)\,{}_9C_6}=\frac{1+8+64}{1+64+4096}=\frac{73}{4161}=\frac{1}{57}$$

30. $(1 + x)^n = c_0 + c_1x + c_2x^2 + \dots + c_nx^n \Rightarrow \int_0^1 (1+x)^n = \int_0^1 (c_0 + c_1x + c_2x^2 + \dots + c_nx^n)dx$

$$\Rightarrow \frac{(1+x)^n}{n+1}\bigg|_0^1=\left[c_0x+\frac{c_1x^2}{2}+\frac{c_2x^3}{3}+\dots+\frac{c_nx^{n+1}}{n+1}\right]_0^1 \Rightarrow \qquad .$$

31. First part is omitted.

$$1 + (2p-1)x + (3p-2)(p-1)\frac{x^2}{2!} + (4p-3)\frac{(p-1)(p-2)}{3!}x^3 + ...$$

$$= 1 + [p + (p-1)]x + [2p + (p-2)](p-1)\frac{x^2}{2!} + [3p + (p-3)]\frac{(p-1)(p-2)}{3!}x^3 + ...$$

$$= \left\{px + 2\frac{p(p-1)}{2!}x + 3\frac{p(p-1)(p-2)}{3!}x^2 + ...\right\} + \left\{1 + (p-1)x + \frac{(p-1)(p-2)}{2!}x^2 + \frac{(p-1)(p-2)(p-3)}{3!}x^3 + ...\right\}$$

$= x\ \{{}_pC_1 + 2\ {}_pC_2\ x + 3\ {}_pC_3\ x^2 + \ldots\} + \{1 + {}_{p-1}C_1\ x + {}_{p-1}C_2\ x^2 + {}_{p-1}C_3\ x^3 + \ldots\}$

$$= x\frac{d}{dx}(1+x)^p + (1+x)^{p-1} = px(1+x)^{p-1} + (1+x)^{p-1} = (px+1)(1+x)^{p-1}$$

32. $(x^2 + 2x + 2)^n = [2 + (2x + x^2)]^n = {}_nC_0\ 2^n + {}_nC_1\ 2^{n-1}\ (2x + x^2) + {}_nC_2\ 2^{n-2}\ (2x + x^2)^2 + {}_nC_3\ 2^{n-3}\ (2x + x^2)^3 + \ldots$

Coeff. of x^2 –term $= {}_nC_1\ 2^{n-1} + {}_nC_2\ 2^{n-2}\ (2^2) = 2^{n-1}\ n^2$.

Coeff. of x^3 –term $= {}_nC_2\ 2^{n-2}\ (2 \times 2) + {}_nC_3\ 2^{n-3}\ (2^3) =$.

33. Coefficientof the term in $x^5 = 21$ and Coefficientof the term in $1/x^5 = 140$.

34. $a = -20$, $b = 200$

35. $(1 + 2x + 3x^2)^n = [(1 + 2x) + 3x^2]^n = {}_nC_0\ (1 + 2x)^n + {}_nC_1\ (1 + 2x)^{n-1}\ (3x^2) + {}_nC_2\ (1 + 2x)^{n-2}\ (3x^2)^2 + \ldots$

Coeff. of x^4 –term $= {}_nC_0\ {}_nC_4\ 2^4 + {}_nC_1\ {}_{n-1}C_2\ 2^2\ 3 + {}_nC_2\ {}_{n-2}C_0\ 3^2$

Coeff. of x^3 –term $= {}_nC_0\ {}_nC_3\ 2^3 + {}_nC_1\ {}_{n-1}C_1\ 2 \times 3$

Coeff. of x^4 –term : Coeff. of x^3 –term $= 121 / 28 \Rightarrow 14\ n^2 - 65n - 376 = 0 \Rightarrow (n - 8)(14n + 47) = 0$

$\therefore\ n = 8$ (negative root is rejected)

36. Put $x = -1$, result follows.

37. $(a + x)^n = c_0\ a^n + c_1\ a^{n-1}x + c_2\ a^{n-2}\ x^2 + \ldots + c_nx^n$.

$(a - x)^n = c_0\ a^n - c_1\ a^{n-1}x + c_2\ a^{n-2}\ x^2 - \ldots + (-1)^n\ c_nx^n$.

$2\ s_1 = (a + x)^n + (a - x)^n$ …. (1) , $2\ s_2 = (a + x)^n - (a - x)^n$ …. (2)

$[(1) + (2)] /2$, $s_1 + s_2 = (a + x)^n$ …. (3) , $[(1) - (2)] /2$, $s_1 - s_2 = (a - x)^n$ …. (4)

$(3) \times (4)$, $s_1^2 - s_2^2 = (a^2 - x^2)^n$, $(1) \times (2)$, $4s_1s_2 = (a + x)^{2n} - (a - x)^{2n}$.

38. $(1 + x)^n = (1 + x)^2\ (1 + x)^{n-2} = (1 + 2x + x^2)\ (1 + x)^{n-2}$

Compare coeff. of x^r –term, ${}_nC_r = {}_{n-1}C_r + 2({}_{n-2}C_{r-1}) + {}_{n-2}C_{r-2}$.

39. Same as 20 (a).

40. $(3 + 2x - x^2)(1+x)^{34} = (3 - x)(1 + x)(1 + x)^{34} = (3 - x)(1 + x)^{35}$.

Coeff. of x^r – term $= 3\ {}_{35}C_{r-1} - {}_{35}C_r = 0$, $1 \le r \le 35$. $\therefore\ 3\ {}_{35}C_{r-1} = {}_{35}C_r \Rightarrow r = 6$.

41. $(1 - x + x^2)^{3n} = a_0 + a_1x + a_2\ x^2 + \ldots$ ….(1) , $(x + 1)^{3n} = c_0x^{3n} + c_1x^{3n-1} + c_2x^{3n-2} + \ldots$ …. (2)

Put $x = 1$ in (1), $a_0 + a_1 + a_2 + \ldots = 1$.

Consider $(x + 1)^{3n}\ (1 - x + x^2)^{3n} = (1 + x^3)^{3n}$

Compare coeff. of x^{3n} –term, $a_0c_0 + a_1c_1 + a_2c_2 + \ldots = {}_{3n}C_n = \dfrac{(3n)!}{n!(2n)!}$.

42. $(1+x)^{2p}(1-x)^p = \left[1+2px+\frac{2p(2p-1)}{2!}x^2+...\right]\left[1-px+\frac{p(p-1)}{2!}x^2+...\right]$

Coeff. of x = coeff. of x^2 $\Rightarrow p=-p^2+p(2p-1)+\frac{p(p-1)}{2}$ $\Rightarrow p(p-5)=0$ $\Rightarrow$ $p=5$

43. Let $a = 3\sqrt{3}$, $b = 5$. Then $I + F = (3\sqrt{3}+5)^{2n+1} = (a+b)^{2n+1}$.

$0 < 3\sqrt{3}-5 < 1$ $\Rightarrow$ $0 < G = (3\sqrt{3}-5)^{2n+1} = (a-b)^{2n+1} < 1$.

$(a+b)^{2n+1} = a^{2n+1} + {}_{2n+1}C_1a^{2n}b + {}_{2n+1}C_2a^{2n-1}b^2 + ... + {}_{2n+1}C_{2n+1}b^{2n+1}$ …. (1)

$(a-b)^{2n+1} = a^{2n+1} - {}_{2n+1}C_1a^{2n}b + {}_{2n+1}C_2a^{2n-1}b^2 - ... - {}_{2n+1}C_{2n+1}b^{2n+1}$ …. (2)

$\therefore$ $(a+b)^{2n+1}-(a-b)^{2n+1}$ contains even powers of a and is an integer.

$\therefore$ $I + F - G$ is an integer. But I is an integer. $\therefore$ $F - G$ is an integer.

Since $0 < F, G < 1$. $\therefore F - G = 0$ $\therefore$ $F = G$.

$\therefore$ $F(I + F) = G(I + F) = (a+b)^{2n+1}(a-b)^{2n+1} = (a^2-b^2)^{2n+1} = \left[(3\sqrt{3})^2-5^2\right]^{2n+1} = 2^{2n+1}$.

44. **(a)** $\sum_{r=0}^{mn} a_r x^r \equiv (1+x+...+x^m)^n$ …. (1)

(i) Put $x = 1$ in (1), $\sum_{r=0}^{mn} a_r = (m+1)^n$

(ii) Put $x = -1$ in (1), $\sum_{r=0}^{mn} (-1)^r a_r = [1-1+...+(-1)^n] = \begin{cases} 1\text{, when m is even} \\ 0\text{ , when m is odd} \end{cases}$

(iii) Put $x = 2$ in (1), $\sum_{r=0}^{mn} 2^r a_r = (1+2+...+2^m)^n = \left(\frac{2^{m+1}-1}{2-1}\right)^n = (2^{m+1}-1)^n$

(b) Differentiate (1), $\sum_{r=0}^{mn} ra_r x^{r-1} \equiv n(1+x+...+x^m)^{n-1}(1+2x+3x^2+...+mx^{m-1})$ …. (2)

Put $x = 1$ in (2), $\sum_{r=0}^{mn} ra_r \equiv n(m+1)^{n-1}\frac{m(m+1)}{2} = \frac{1}{2}mn(m+1)$

$\therefore$ $mn\sum_{r=0}^{mn} a_r = 2\sum_{r=0}^{mn} ra_r$.

(c) When m is even, $m = 2p$.

$\sum_{r=1}^{m}(-1)^r r = -1+2-3+4-...+2p = (-1+2)+(-3+4)+...+[-(2p-1)+2p] = p = \frac{m}{2}$

When m is odd, $m = 2p + 1$.

$\sum_{r=1}^{m}(-1)^r r = -1+2-3+4-...-(2p+1) = (-1+2)+(-3+4)+...+[-(2p-1)+2p]-(2p+1)$

$= p-(2p+1) = -p-1 = -\frac{m+1}{2}$

Put $x = -1$ in (2), $\sum_{r=1}^{m}(-1)^{r-1} ra_r = n[1-1+...+(-1)^m]^{n-1}\left[\sum_{r=1}^{mn}(-1)^r r\right]$

$\therefore$ $2\sum_{r=1}^{m}(-1)^r ra_r = \begin{cases} mn & \text{,if } m \text{ is even} \\ 0 & \text{,if } m \text{ is odd.} \end{cases}$

45. ${}_{n}C_{r+1} = {}_{n-1}C_r + {}_{n-1}C_{r+1}$. omitted – bookwork.

$$\begin{aligned}
{}_{n}C_r &= {}_{n-1}C_r &&+ {}_{n-1}C_{r-1}\\
{}_{n-1}C_{r-1} &= {}_{n-2}C_{r-1} &&+ {}_{n-2}C_{r-2}\\
{}_{n-2}C_{r-2} &= {}_{n-3}C_{r-2} &&+ {}_{n-3}C_{r-3}\\
&\vdots &&\vdots\\
{}_{n-r+1}C_1 &= {}_{n-r-2}C_1 &&+ {}_{n-r-2}C_0
\end{aligned}$$

Adding all equalities and cancel terms, put ${}_{n-r-2}C_0$ to ${}_{n-r-1}C_0$. Result follows.

46. ${}_{n}C_r = {}_{n-1}C_{r-1} + {}_{n-1}C_r$. omitted – bookwork.

P(n) : $\displaystyle\sum_{q=0}^{n} {}_{n+q}C_q \frac{1}{2^{n+q}} = 1$.

For P(0), L.H.S. = ${}_{0}C_0\left(\frac{1}{2^0}\right) = 1$ = R.H.S. $\therefore$ P(1) is true.

Assume P(k) is true for some $k \in \mathbf{N}$, i.e. $\displaystyle\sum_{q=0}^{k} {}_{k+q}C_q \frac{1}{2^{k+q}} = 1$ …. (*)

$$s = \sum_{q=0}^{k+1} {}_{k+1+q}C_q \frac{1}{2^{k+1+q}} = \frac{1}{2^{k+1}} + \sum_{q=1}^{k+1}\left({}_{k+q}C_q + {}_{k+q}C_{q-1}\right)\frac{1}{2^{k+1+q}} = \frac{1}{2^{k+1}} + \frac{1}{2}\sum_{q=1}^{k+1} {}_{k+q}C_q \frac{1}{2^{k+q}} + \sum_{q=1}^{k+1} {}_{k+q}C_{q-1}\frac{1}{2^{k+1+q}}$$

$$= \frac{1}{2^{k+1}} + \frac{1}{2}\left[\sum_{q=0}^{k+1} {}_{k+q}C_q \frac{1}{2^{k+q}} - \frac{1}{2^k} + {}_{2k+1}C_{k+1}\frac{1}{2^{2k+1}}\right] + \sum_{q=0}^{k} {}_{k+q+1}C_q \frac{1}{2^{k+q+2}}$$

$$= \frac{1}{2}\left[1 + {}_{2k+1}C_{k+1}\frac{1}{2^{2k+1}}\right] + \frac{1}{2}\left[s - {}_{2k+1}C_{k+2}\frac{1}{2^{2k+2}}\right] \quad \text{, by (*)}$$

$$= \frac{1}{2} + \frac{1}{2}s, \qquad \because\ {}_{2k+1}C_{k+1}\frac{1}{2^{2k+1}} = {}_{2k+1}C_{k+2}\frac{1}{2^{2k+2}}$$

$\therefore$ s = 1 and P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall\, n \in \mathbf{N}$.

47. **(a)** (Case 1) For $n \ge r$. If n is a positive integer, use Math. Induction to prove the assertion.

Use $C_r(n + 1) = C_r(n) + C_{r-1}(n)$ to complete the assertion. (sum of two integers is an integer)

Also, if n = 0 , $C_r(0) = 0$.

(Case 2) For $0 \le n < r$, One of the factor n , (n – 1), …., (n – r + 1) is zero. $\therefore$ $C_r(n) = 0 \in \mathbf{Z}$.

(Case 3) If $n \in -\mathbf{Z}$, put m = -n,

$$C_r(n) = \frac{n(n-1)\ldots(n-r+1)}{r!} = \frac{(-m)(-m-1)\ldots(-m-r+1)}{r!} = (-1)^r\,\frac{m(m+1)(m+2)\ldots(m+r-1)}{r!}$$

$= (-1)^r C_r(m + r – 1)$ which is an integer by Case 1 .

(b) deg [p(x)] = k, deg $[C_i(x)] = i$ where $0 \le i \le k$.

By division algorithm, we can get

$p(x) = b_k C_k(x) + r_k(x)$, $\deg[r_k(x)] = k-1 < \deg[C_k(x)] = k$

$r_k(x) = b_{k-1} C_k(x) + r_{k-1}(x)$, $\deg[r_{k-1}(x)] = k-2$

: : : :

$r_2(x) = b_1 C_1(x) + r_1(x)$, $\deg[r_1(x)] = 0$

$r_1(x) = b_0 C_0(x)$.

Adding all equalities, $p(x) = b_k C_k(x) + b_{k-1} C_{k-1}(x) + \ldots + b_0 C_0(x)$ …. (1)

Now, $p(0) = b_0 C_0(0) \in \mathbf{Z}$ since $b_0, C_0(x) \in \mathbf{Z}$.

$p(1) = b_1 C_1(1) + b_0 C_0(1) \in \mathbf{Z}$ since $b_1, C_1(1), b_0, C_0(1) \in \mathbf{Z}$.

Similarly, $p(2) = b_2 C_1(2) + b_0 C_0(2) + b_0 C_0(2) \in \mathbf{Z}$ **.**

Continue in this way, all b_i's in (1) are integers.

(c) Similar to (b), instead of $\mathbf{Z}$, we use $\mathbf{Q}$, we can show that $b_i \in \mathbf{Q}$.

By (1), all coefficients of $p(x)$ are rationals.

48. **(a)** Same as 21. (a) .

(b) $C_k^n \dfrac{1}{n^k} = \dfrac{n(n-1)(n-2)....(n-k+1)}{k!}\dfrac{1}{n^k} = \dfrac{1}{k!}\left(1-\dfrac{1}{n}\right)\left(1-\dfrac{2}{n}\right)...\left(1-\dfrac{k-1}{n}\right) \le \dfrac{1}{k!}$ since all factors in brackets ≤ 1

(c) $\left(1+\dfrac{1}{n}\right)^n = \sum_{k=0}^{n} {}_nC_k \dfrac{1}{n^k} \underset{\text{by (b)}}{\le} \sum_{k=0}^{n} \dfrac{1}{k!} = 1+1+\dfrac{1}{2!}+\dfrac{1}{3!}+...+\dfrac{1}{n!} < 1+1+\dfrac{1}{2}+\dfrac{1}{2^2}+...+\dfrac{1}{2^n}$ (Geometric series)

$< 1+\left(1+\dfrac{1}{2}+\dfrac{1}{2^2}+...+\dfrac{1}{2^n}+...\right)$ (Infinite geometric series) $= 1 + \dfrac{1}{1-\dfrac{1}{2}} = 3.$

49. **(a)** If $m-p \ge p$, $G(m, m-p)$

$$= \frac{(1-x^m)(1-x^{m-1})...(1-x^{m-p+1})(1-x^{m-p})(1-x^{m-p-1})...(1-x^{p+1})}{(1-x)(1-x^2)...(1-x^p)(1-x^{p+1})...(1-x^{m-p-1})(1-x^{m-p})} = \frac{(1-x^m)(1-x^{m-1})...(1-x^{m-p+1})}{(1-x)(1-x^2)...(1-x^p)} = G(m,p)$$

If $m-p \le p$, we may put $m-p = q$. Then $p = m-q$ and it follows that $m-q \ge q$. From above,

$G(m, m-p) = G(m, q) = G(m, m-q) = G(m, p)$.

(b) **(i)** $G(m, p+1) - G(m-1, p+1)$

$$= \frac{(1-x^m)(1-x^{m-1})...(1-x^{m-p})}{(1-x)(1-x^2)...(1-x^{p+1})} - \frac{(1-x^m)(1-x^{m-1})...(1-x^{m-p-1})}{(1-x)(1-x^2)...(1-x^{p+1})}$$

$$= \left[(1-x^m)-(1-x^{m-p-1})\right]\frac{(1-x^m)(1-x^{m-1})...(1-x^{m-p})}{(1-x)(1-x^2)...(1-x^{p+1})} = x^{m-p-1}(1-x^{p+1})G(m-1,p)\frac{1}{1-x^{p+1}}$$

$= x^{m-p-1} G(m-1, p)$.

(ii) From (b) (i) , we get: $G(n, p+1) = G(n-1, p+1) + x^{n-p-1} G(n-1, p)$.

Putting $p+1, p+2, \ldots, m$ for n ,

$G(p + 1, p + 1) = G(p, p) + G(p, p + 1)$

$G(p + 2, p + 1) = xG(p + 1, p) + G(p + 1, p + 1)$

$G(p + 3, p + 1) = x^2 G(p + 2, p) + G(p + 2, p + 1)$

: : : : : : :

$G(m, p + 1) = x^{m-p-1} G(m - 1, p) + G(m - 1, p + 1)$

Adding all these equalities, we have :

$G(m, p + 1) = G(p, p) + G(p, p + 1) + xG(p + 1, p) + x^2 G(p + 2, p) + \dots + x^{m-p-1} G(m - 1, p)$

But $G(p, p + 1) = 0$, result follows.

(c) For $p = 1$, if $m = 0$, then $G(m, p) = G(0, 1) = \frac{1-x^0}{1-x} = 0$

If $m \geq 1$, then $G(m, p) = \frac{1-x^m}{1-x} = 1 + x + x^2 + \dots + x^{m-1}$.

Thus, when $p = 1$, $G(m, p)$ is a polynomial in x for any integer $m \geq p - 1$.

Assume that $G(m, p)$ is a polynomial in x for any $m \geq p - 1$.

Thus when $m \geq p - 1$, $G(p, p), G(p + 1, p), G(p + 2, p), \dots , G(m - 1, p)$ are all polynomials in x .

Therefore by (b) (ii) , $G(m, p +1)$ is also a polynomial in x .

Moreover, we have shown that $G(p, p + 1)$ is a polynomial in x for any positive integer $m \geq p$.

So, by the Principle of Induction, $G(m, p)$ is a polynomial in x for any $m, p \in \mathbf{N}, m \geq p - 1$.

50. **(a)** $f'(a_i) = (a_i - a_1)(a_i - a_2)\dots(a_i - a_{i-1})(a_i - a_{i+1})\dots(a_i - a_n)$

(b) **(i)** (*)

$\therefore$ $g(a_i) = A_i (a_i - a_1)(a_i - a_2)\dots(a_i - a_{i-1})(a_i - a_{i+1})\dots(a_i - a_n) = A_i f'(a_i)$.

$\therefore$ $A_i = \frac{g(a_i)}{f'(a_i)}$

(ii) If $\deg [g(x)] < n - 1$, the coefficient of x^n of $g(x) = 0$ and so $\sum_{i=1}^{n} A_i = 0$.

$\therefore$

(iii) Let $g(x) = x^m$ where $m \leq n - 2$. Then $f(x) = (x - 1)(x - 2)\dots(x - n)$.

$f'(i) = (i - 1)\dots[i - (i - 1)][i - (i + 1)\dots(i - n) = (i - 1)!\,(n - i)!\,(-1)^{n-i}$.

By (ii), $\sum_{i=1}^{n} (-1)^{n-i} \frac{i^m}{(i-1)!(n-i)!} = 0$